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were balloted for and duly elected Associates of the Society.

On the General Solution of the Problem of Disturbed Elliptic Motion. By E. Neison, Esq.

There are two distinct methods of dealing with the problem of disturbed elliptic motion. One method is to throw the effect of the perturbations directly on the three coordinates, or to suppose that the disturbed body moves in an irregular curve which is

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approximately of the form of an ellipse. The other method is to throw the perturbations directly on the elements of the elliptic orbit, by assuming the body to move in an ellipse which is perpetually changing in form and position. The first method may be called that of perturbed coordinates, and has the advantage of requiring only three variables, but in turn the disadvantage of employing more complex differential equations. The second method, which may be termed that of perturbed elements, has the disadvantage of requiring six or seven variables, but requires a very much simpler form of differential equations.

What is required for astronomical purposes is the polar coordinates of the disturbed body. By making these polar coordinates the three variables of the first method, their values at any instant is at once obtained. But by the second method what is obtained is only the value of the elements of the ellipse in which the disturbed body is momentarily moving. These must be transformed into an expression for the required polar coordinates, and this necessitates extra work. Consequently there is this additional disadvantage in the method of perturbed elements.

Despite the disadvantages which have been pointed out, the method of perturbed elements is the simplest and most expeditious way of determining the perturbations of the order of the first power of the disturbing forces. For this reason it is admirably adapted for employment in the planetary theories, where powers of the disturbing forces are rarely required. In the lunar theory, however, it is absolutely essential to consider the square of the disturbing forces. But this renders most complex the known methods of determining the perturbations of the elements and necessitates an enormous increase in the labour. For instead of there being only six powers and products of the three variable coordinates, the second method involves the formation of no less than twenty-eight powers and products of the seven variable elements. The calculation of these powers and products is laborious and difficult. Therefore, under these conditions, the second method involves many times as much work as the first method.

For these reasons the earlier workers on the lunar theory —Damoiseau, Plana, Lubbock, Pontécoulant—employed the method of perturbed coordinates. The polar coordinates were taken for variables, and the differential equations were simplified by omitting all quantities of a higher order than it was intended to retain in the approximation, whilst by special artifices the amount of work was lessened. Poisson strongly advocated the method of perturbed elements, and, at his instigation, Pontécoulant employed it in certain parts of his lunar theory, but pointed out that to calculate the lunar inequalities by this method would be a work of immense labour. Hansen made use of a method which is really a modification of the method of perturbed coordinates, though he did not throw the perturbations directly

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on the coordinates. In other respects his system of using his method was the same as that employed by his predecessors. In all these cases, therefore, special means were taken to obtain a solution which should be complete to a given order, but which could not be extended to any further order, and depended entirely on the special features of the problem. Lately, Hill, by making use of rectangular coordinates and a modified form of the differential equations, has successfully determined the value of the perturbations which are independent of the eccentricities. His method possesses some generality of form, and is not limited to the particular problem. It consists of an expression connecting together the different indeterminate coefficients. It is, however, an expression which gives a very complex form to the complete coefficient of the *n*th term, and is restricted to the coefficient of a small class of easy terms. Moreover, it yields the value of the rectangular coordinates and requires extra work to transform it into an expression for the polar coordinates. It is, however, to be remarked that for the definite purpose which its author had in view these are matters of indifference.

Delaunay, alone, has successfully made use of a method of throwing the perturbations on the elements of the elliptic orbit. He does not make use of the ordinary method of the variations of elliptic elements—that is, the method which has been employed by Pontécoulant and Le Verrier in the planetary theories, or which was advocated by Poisson for use in the lunar theory. Delaunay devised a most ingenious method, which really consists in determining the value of the elements which will make the disturbing function assume a given form ; a form which enables him to directly integrate the resulting differential equations. It is a most elegant and ingenious method, but is far more complex than any other, and involves an enormous amount of work. It is no exaggeration to say that it involves thrice the amount of work which would have been necessary to obtain the same result by the use of Pontécoulant's method. It is, moreover, purely a special method, a method depending on a long series of particular transformations, and it would be impracticable to obtain by its means a general expression for the perturbations of the coordinates or elements. On the other hand, Delaunay's method has an important advantage over any method involving the use of indeterminate coefficients, as Hill's or my own ; it is possible to separately determine the value of each single term in the perturbations without its being necessary to determine any other.

It has long been my opinion that, by choosing suitable variables, and by transforming the differential equations in a proper manner, it would be possible to obtain a complete general solution of the problem of disturbed elliptic motion ; that is to say, a solution which would be so far general that it would be possible to write down the perturbations to the *n*th order of the disturbing forces, and be independent of any particular values of

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the elements. Such a solution would be obtained if it were possible to form a simple expression for the complete perturbations of the order of the nth power of the disturbing forces. It seemed to me that if such an expression could be formed, it would be an important contribution to the theory of disturbed elliptic motion, and in particular to the lunar theory. For several years, therefore, it has been a problem which has engaged much of my attention.

It was not long before it was seen that though it would be possible to obtain an expression for the perturbations of the order of the nth power of the disturbing forces, yet it was evident that it would be very difficult to prevent this expression becoming very lengthy and complex. This difficulty has now been overcome, and a very simple expression has been found for the perturbations of the order of the nth power of the disturbing forces.

In the present paper I propose to give a short account of the principal results, sufficiently full to enable them to be understood; but the details and minor developments are deferred to a subsequent communication, when I propose to apply the results to some important points in the lunar theory.

Let $a\mathrm{R}$ be the disturbing function when the elements of the orbit have their instantaneous values x, y, z, X, Y, Z, the coefficient a being supposed to be a constant, and let the general term of R be denoted by

$$\mathrm{R}\cos \mathrm{w} = (\Lambda)\, \mathrm{x}^{\mathrm{F}} \mathrm{y}^{\mathrm{G}} \mathrm{z}^{\mathrm{H}} \cos\{(\lambda) + f\mathrm{X} + g\mathrm{Y} + h\mathrm{Z}\}, \tag{1}$$

where (Λ) and (λ) are supposed to be independent of the elements of the orbit, and to be absolutely constant.

Suppose R to denote the value of R when the elements have their constant mean values. Then R will be the same function of the mean elements, x, y, z, X, Y, Z, as R is of the instantaneous elements, x, y, z, X, Y, Z, so that the general term of R can be written in the form

$$R\cos w = (\Lambda)\, x^{\mathrm{F}} y^{\mathrm{G}} z^{\mathrm{H}} \cos\{(\lambda) + fX + gY + hZ\}. \tag{2}$$

Let each element of the orbit be considered as the sum of its constant mean value, and either of a single periodical function which represents its perturbations, or of n periodical functions which collectively represent its perturbations, so that, for instance,

$$\mathrm{x} = x + \delta x = x + \delta_1 x + \delta_2 x + \ldots\ldots + \delta_{\mathrm{n}} x = x + \Sigma\delta_m x. \tag{3}$$

Then R is to be developed in terms of R and these variables, which represent the perturbations of the six elements.

Let v be taken as the general representative of the three elements which have been denoted by x, y, z, and I for the representative of the corresponding coefficients F, G, H; and similarly suppose V to represent the elements which have been denoted by

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X, Y, Z and i to represent the corresponding coefficients f, g, h. Then it may be supposed that

$$\mathrm{v} = v + \delta v = v + \Sigma \delta_m v, \qquad \mathrm{V} = V + \delta V = V + \Sigma \delta_m V, \tag{4}$$

where δv, δV represent the periodical perturbations of the elements whose mean values are represented by v and V. Assume further that

$$\begin{aligned} v^{-1}\delta v &= \Sigma c_p \cos \gamma_p, & \delta V &= \Sigma \mathrm{C}_p \sin \gamma_p, \\ v^{-1}\delta_m v &= \Sigma j_p^m \frac{1}{w_p} R_p^m \cos w_p, & \delta_m V &= \Sigma \mathrm{J}_p^m \frac{1}{w_p} R_p^m \sin w_p, \end{aligned} \tag{5}$$

where m denotes the number of accents to be attached to j and J.

Next by successive direct differentiation

$$v^s \frac{d^s R}{dv^s} = \Sigma \frac{\Gamma(\mathrm{I})}{\Gamma(\mathrm{I}-s)} R \cos w, \qquad \frac{d^s R}{dV^s} = \Sigma i^s \{\cos s \frac{\pi}{2} . R \cos w - \sin s \frac{\pi}{2} . R \sin w\}. \tag{6}$$

Put

$$\mathrm{D}_v = [v^{-1}\delta v]\, v \frac{d}{dv}, \qquad \mathrm{D}_V = \delta V \frac{d}{dV},$$

then a known form of expansion gives the expression

$$\begin{aligned} \mathrm{R} &= R + \delta R = \{\epsilon^{\Sigma(\mathrm{D}_v + \mathrm{D}_V)}\} R \\ &= \{\Sigma (1 + \mathrm{D}_v + \frac{1}{\underline{|2}} [\mathrm{D}_v]^2 + \dots)(1 + \mathrm{D}_V + \frac{1}{\underline{|2}} [\mathrm{D}_V]^2 + \dots) \dots \} R. \end{aligned} \tag{7}$$

From the value assigned to D the following expressions can be derived:—

$$\begin{aligned} [\mathrm{D}_v]^n R \cos w &= \left(\frac{1}{2}\right)^n \{\mathrm{I}(\mathrm{I}-1)\dots(\mathrm{I}-n+1)\}[c_1 + c_2 + c_3 + \dots + c_n]^n \\ &\qquad \cos(w + q'\gamma_1 + q''\gamma_2 + \dots + q^n\gamma_n), \\ [\mathrm{D}_V]^n R \cos w &= \left(\frac{1}{2}\right)^n \{q'q'' \dots q^n\}(i)^n \quad [\mathrm{C}_1 + \mathrm{C}_2 + \mathrm{C}_3 + \dots + \mathrm{C}_n]^n \\ &\qquad \cos(w + q'\gamma_1 + q''\gamma_2 + \dots + q^n\gamma_n), \\ [\mathrm{D}_v]^m [\mathrm{D}_V]^{m-n} R \cos w &= \left(\frac{1}{2}\right)^n \{q^m q^{m+1} \dots q^n\}(i)^{n-m}\{\mathrm{I}(\mathrm{I}-1)\dots(\mathrm{I}-m+1)\} \\ &\qquad [c_1 + c_2 + \dots + c_n]^m [\mathrm{C}_m + \mathrm{C}_{m+1} + \dots + \mathrm{C}_m]^{n-m} \cos(w + q'\gamma_1 + q''\gamma_2 + \dots + q^n\gamma_n). \end{aligned} \tag{8}$$

Then putting for brevity

$$\begin{aligned} (\mathrm{I}c)^m &= \{\mathrm{I}(\mathrm{I}-1)\dots(\mathrm{I}-m+1)\}[c_1 + c_2 + \dots + c_m]^m, \\ (qi\mathrm{C})^m &= (i)^m \{q'q'' \dots q^m\} \quad [\mathrm{C}_1 + \mathrm{C}_2 + \dots + \mathrm{C}_m]^m, \end{aligned} \tag{9}$$

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the complete value of the general term of R will be

$$\mathrm{R}\cos \mathrm{w} = R\cos w + \frac{1}{2} R\,(\mathrm{I}c_1 + q' i \mathrm{C}_1)\cos(w + q'\gamma_1) + \ldots$$
$$+ \frac{1}{\lfloor n}\left(\frac{1}{2}\right)^n R\,(\mathrm{I}c + qi\mathrm{C})^n \cos(w + \Sigma_1^{\,n} q^m \gamma_m) + \ldots \qquad (10)$$

In the preceding expressions the coefficients $q'\ q'' \ldots q^m$ only take the values plus or minus unity, and have been introduced for the purpose of clearly showing the changes of sign produced by changes of sign in the argument.

Let

$$\mathrm{R} = \Sigma\{R_p + \mathrm{A}_p\}\cos w_p, \qquad (11)$$

where A_p denotes the sum of the coefficients of the argument w_p which arises from the portion of R depending on the perturbations, so that $R_p + \mathrm{A}_p$ denotes the complete value of the coefficient of the argument w_p in the value of R. The function A is therefore a constant, and a function of R_p, the coefficients I, i, q, and the indeterminate coefficients c and C.

Let (R) be a function of similar form to R, and such that

$$(\mathrm{R}) = \mathrm{x}^{(\mathrm{F})}\mathrm{y}^{(\mathrm{G})}\mathrm{z}^{(\mathrm{H})}\,\mathrm{R}, \qquad (12)$$

so that the general term of (R) may be written in the form

$$(\mathrm{R})\cos \mathrm{w} = (\Lambda)\ \mathrm{x}^{\mathrm{F}+(\mathrm{F})}\mathrm{y}^{\mathrm{G}+(\mathrm{G})}\mathrm{z}^{\mathrm{H}+(\mathrm{H})}\cos\{(\lambda) + f\mathrm{X} + g\mathrm{Y} + h\mathrm{Z}\}, \qquad (13)$$

and is evidently merely a particular case of the general term (1). Its expansion will be identical in form with that of (10). Suppose therefore,

$$(\mathrm{R}) = \Sigma x^{(\mathrm{F})} y^{(\mathrm{G})} z^{(\mathrm{H})}\{R_p + (\mathrm{A})_p\}\cos w_p, \qquad (14)$$

where $(\mathrm{A})_p$ only differs from A_p in F, G, H being replaced by F+(F), G+(G), H+(H) respectively; so that $(\mathrm{A})_p$ is merely a particular case of A_p.

Suppose the coefficients of the perturbations in the six elements of the orbit to be represented by the $6k$ indeterminate coefficients $c_1 \ldots c_{6k}$, then the function A_p can be written in the form

$$\mathrm{A}_p = \Sigma\phi(r)\,c_s + \Sigma\phi'(r)\,(c_s c_{s_1}) + \Sigma\phi''(r)\,(c_s c_{s_1} c_{s_2}) + \ldots + \Sigma\phi^n(r)\,(c_s \ldots c_{s_n}) + \ldots, \qquad (15)$$

where $\phi(r), \ldots \phi^n(r)$ are known constants, functions of R_r, I_r, i_r, and $c_s, \ldots c_{s_n}$ are particular cases of the series of indeterminate coefficients which represent the perturbations of the elements. The summation extends to all the combinations of the suffixes r and $s, \ldots s_n$ which will yield a term with the argument w_p.

By means of (10) the value of the coefficient A_p as developed in (15) can be written down to the nth order. It is merely

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necessary to know the different combinations of the arguments $w_r, \gamma_1, \ldots \gamma_k$ which will render $w_r + q\gamma_1 + \ldots + q^m\gamma_m$ equal to the value w_p. Suppose there are l such independent combinations of $w_r, \gamma_1, \ldots \gamma_k$, then by giving to $s, \ldots s_m$ and r these l series of values, the complete value of $\Sigma\phi^m(r)(c_s \ldots, c_{s_m})$ as it exists in A_p will be obtained. It only remains therefore to determine these l combinations of the arguments. For this purpose let

$$R = \Sigma_0^k R_m \cos w_m.$$

By direct successive multiplication form the functions

$$\{R\}^2, \{R\}^3, \ldots \{R\}^n,$$

neglecting terms of a higher order than necessary for an approximation to the given order. Then the different combinations of m in the value of the coefficient of the argument w_p in the function $\{R\}^m$ will be the l combinations of $r, s_1, \ldots s_n$ in the value of $\Sigma\phi^m(r)(c_s \ldots c_{s_m})$ as it exists in the coefficient A_p.

In this way the value of A_p can be completely determined to the nth order, or the $2n$th order as usually computed, so that R may be regarded as completely determined to that order.

Suppose the system of elements represented by v, V to be a canonical system, so that the differential equations which give the perturbations in the elements may be written in the form

$$\frac{d\mathrm{v}}{dt} = \frac{d\mathrm{R}}{d\mathrm{V}} = -\Sigma i\mathrm{R}\sin\mathrm{w}, \qquad \frac{d\mathrm{V}}{dt} = \frac{d\mathrm{R}}{d\mathrm{v}} = \mathrm{v}^{-1}\Sigma\mathrm{IR}\cos\mathrm{w}. \qquad (16)$$

Let it be assumed that

$$\int {}^{\cos}_{\sin}(w)\,dt = \pm\frac{1}{w}{}^{\sin}_{\cos}(w),$$

and integrate the system of equations (16). They become

$$v^{-1}\delta v = \Sigma\frac{i}{v}\frac{1}{w}\{R + \delta R\}\cos w, \qquad \delta V = \Sigma\frac{1}{v}\frac{1}{w}\{R + (\delta R)\sin w\}. \quad (17)$$

Therefore

$$\left.\begin{aligned} c_p\cos w_p &= \{i_p\frac{1}{v}\frac{1}{w_p}\,R_p + i_r\frac{1}{v}\frac{1}{w_p}\,A_p\,\}\cos w_p, \\ C_p\sin w_p &= \{I_p\frac{1}{v}\frac{1}{w_p}\,R_p + I_r\frac{1}{v}\frac{1}{w_p}\,(A)_p\}\sin w_p. \end{aligned}\right\} \quad (18)$$

In this manner, amongst the $6k$ coefficients $c_1, \ldots c_{6k}$, there will be obtained $6k$ independent equations of the form

$$c_p = \frac{1}{v}\frac{1}{w_p}\{i_pR_p + \Sigma i_r\,\phi'(r)\,c_s + \ldots + \Sigma i_r\,\phi^n(r)\,(c_s \ldots c_{s_n}) + \ldots\}, \quad (19)$$

where, for brevity, I is included under i and C under c.

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This system of $6k$ equations amongst the $6k$ indeterminate coefficients can be completely solved without difficulty and the values of the $6k$ coefficients $c_1, \ldots c_{6k}$ can be determined to any required order.

It would not be difficult to transform the preceding equations into others of a still more advantageous form, but this is unnecessary at present. In practice it would be found essential to have seven series of unknown coefficients, the extra one being required for the perturbations of the mean motion. Therefore, although the preceding system of equations is very simple, and can be written down without difficulty to any required order, it still retains the serious disadvantage of requiring seven variables. When indeterminate coefficients are employed, this multiplicity of variables is the great difficulty of the method of perturbed elements, for it involves the solution of a system of $7k$ equations amongst $7k$ unknown quantities, whereas, when the coordinates are made the direct variables, as in my "General Method of Treating the Lunar Theory," it is only necessary to solve a system of $3k$ equations amongst $3k$ unknown quantities, and this requires far less labour. Moreover, the preceding method requires the entire system of perturbations to be attacked at once, and it is not practicable to determine the value of any one coefficient by itself. It remains therefore to remove these disadvantages by further developing the method.

Instead of supposing R to be developed in terms of the variables represented by δv and δV, let it be developed in terms of the n times as many variables represented by $\delta_1 v, \ldots \delta_n v$ and $\delta_1 V, \ldots \delta_n V$, and denoted generally by $\delta_m v$ and $\delta_m V$. As the functions represented by $\delta_m v$ and $\delta_m V$ are perfectly analogous to those represented by δv and δV, the form of the expansion will not be altered, and it will merely be necessary to make some slight changes in the notation. Put, then,

$$\left[R_s^m\,(\mathrm{I}_r\,\dot{j}_s^m)\,\frac{1}{w_s}\right]^k = \{\mathrm{I}_r\,(\mathrm{I}_r-1)\ldots(\mathrm{I}_r-k+1)\}\left(\dot{j}_{s_1}^m \mathrm{R}_{s_1}^m \frac{1}{w_{s_1}}+\ldots+\dot{j}_{s_k}^m R_{s_k}^m \frac{1}{w_{s_k}}\right)^k$$

$$\left[R_s^m\,(q i_r \mathrm{J}_s^m)\,\frac{1}{w_s}\right]^k = \{q'q''\ldots q^k\}\,(i)^k \qquad \left(\mathrm{J}_{s_1}^m R_{s_1}^m \frac{1}{w_{s_1}}+\ldots+\mathrm{J}_{s_k}^m R_{s_k}^m \frac{1}{w_{s_k}}\right)^k \tag{20}$$

and let Σ denote a summation with respect to m, so that, for example,

$$\left[\Sigma R_s^m\,(\mathrm{I}_r \dot{j}_s^m)\,\frac{1}{w_s}\right]^k = \{\mathrm{I}_r\,(\mathrm{I}_r-1)\ldots(\mathrm{I}_r-k+1)\}$$

$$\left([j'_{s_1}R'_{s_1}+\ldots+\dot{j}_{s_1}^{\mathrm{n}}R_{s_1}^{\mathrm{n}}]\,\frac{1}{w_{s_1}}\ldots+[j'_{s_k}R'_{s_k}+\ldots+\dot{j}_{s_k}^{\mathrm{n}}R_{s_k}^{\mathrm{n}}]\,\frac{1}{w_{s_k}}\right)^k \tag{21}$$

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Then the general term of R can be written in the form

$$\mathrm{R}\cos \mathrm{w} = R_r \cos w_r + \frac{1}{2}\left[\Sigma R^m_{s_1}\left(\mathrm{I}_r j^m_{s_1} + q' i_r \mathrm{J}^m_{s_1}\right)\frac{1}{w_{s_1}}\right] R_r \cos\left(w_r + q' w_{s_1}\right) + \ldots$$

$$+ \frac{1}{\underline{|n}}\left(\frac{1}{2}\right)^n\left[\Sigma R^m_s\left(\mathrm{I}_r j^m_s + q i_r \mathrm{J}^m_s\right)\frac{1}{w_s}\right]^n R_r \cos\left(w_r + \Sigma^n_1 q w_s\right) + \ldots \qquad (22)$$

Substitute this value with that of (R) in the differential equations, and on integration they become

$$v^{-1}\delta v = \frac{1}{v} i_r \frac{1}{w_r} R_r \cos w_r + \ldots + \frac{1}{v} i_r \frac{1}{w_r + \Sigma^n_1 q w_s} \times$$

$$\frac{1}{\underline{|n}}\left(\frac{1}{2}\right)^n\left[\Sigma R^m_s\left(\mathrm{I}_r j^n_s + q i_r \mathrm{J}^m_s\right) \qquad \frac{1}{w_s}\right]^n R_r \cos\left(w_r + \Sigma^n_1 q w_s\right) + \ldots$$

$$(23)$$

$$\delta V = \frac{1}{v}\mathrm{I}_r \frac{1}{w_r} R_r \sin w_r + \ldots + \frac{1}{v}\mathrm{I}_r \frac{1}{w_r + \Sigma^n_1 q w_s} \times$$

$$\frac{1}{\underline{|n}}\left(\frac{1}{2}\right)^n\left[\Sigma R^m_s\left(\mathrm{I}_r j^m_s + q i_r \mathrm{J}^m_s + (\mathrm{I})\, j^m_s\right)\frac{1}{w_s}\right]^n R_r \sin\left(w_r + \Sigma^n_1 q w_s\right) + \ldots$$

where, as before, the transformation of R into (R) is effected by the addition of the quantity (I), where (I) is an absolutely constant quantity, which is the same for all values of r and s.

Assume that the perturbations of the elements of the order of the mth power of the disturbing forces are represented by the expressions

$$v^{-1}\delta_m v = j^m \frac{1}{w} R^m \cos w, \qquad \delta_m V = \mathrm{J}^m \frac{1}{w} R^m \sin w, \qquad (24)$$

then it only remains to determine the value of the coefficients $j' \ldots j^n$ and $\mathrm{J}' \ldots \mathrm{J}^n$ and the quantities $R' \ldots R^n$, and the right hand side of the equations (23) will become an explicit function of known quantities, so that the complete value of the perturbations of the elements will have been determined to the nth order of the disturbing forces.

Neglecting the square of the disturbing forces, the equations (22) can be written in the form

$$v^{-1}\delta_1 v = \frac{1}{v} i_p \frac{1}{w_p} R_p \cos w_p, \qquad \delta_1 V = \frac{1}{v}\mathrm{I}_p \frac{1}{w_p} R_p \sin w_p, \qquad (25)$$

where the suffix r has been replaced by p. Comparing with (24),

$$j'_p = \frac{1}{v} i_p, \qquad \mathrm{J}'_p = \frac{1}{v}\mathrm{I}_p, \qquad R'_p = R_p. \qquad (26)$$

Thus all the terms depending on j', J', R' may be regarded as known.

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As far as these terms are concerned, the complete value of the term in the disturbing function which has the argument w_k may be written in the form

$$\mathrm{R}\cos\mathrm{w} = R_k \cos w_k + \left[\frac{1}{2} R_{s_1} (\mathrm{I}_r i_{s_1} + q' i_r \mathrm{I}_{s_1}) \frac{1}{v} \frac{1}{w_{s_1}}\right] R_r \cos(w_r + q' w_{s_1}) + \ldots$$

$$+ \frac{1}{\underline{|n}} \left[\frac{1}{2} R_s (\mathrm{I}_r i_s + q i_r \mathrm{J}_s) \frac{1}{v} \frac{1}{w_s}\right]^n R_r \cos(w_r + \Sigma_1^n q w_s) + \ldots \qquad (27)$$

with the single condition that the suffixes r, s_1, s_2, ... s_n which have been used to distinguish quantities belonging to different arguments are to be given in turn all the different combinations of values which will make $w_r + q' w_s + \ldots + q^n w_{s_n}$ equal to w_k. These values will be identical with the l different combinations of the suffixes m in the value of the coefficient of the term with the argument w_k in the function $\{R\}^n$ or nth power of R.

The coefficients j'', ... j^n and J'', ... J^n can be determined in exactly the same manner, and will give rise to similar expression, the principal difference being that the simple coefficients i_s, I_s will be replaced by more complex functions. For the present it is unnecessary to write down these terms seriatim, although it can be at once done from equations (23). The general expression for the terms of the perturbations of the elements of the nth order of the disturbing forces will be

$$v^{-1}\delta_n v = \Sigma j'_r \frac{1}{w_p} R_r \frac{1}{\underline{|b_1} \ldots \underline{|b_m}}$$

$$\left\{\left[\frac{1}{2} R_s (\mathrm{I}_r j'_s + q i_r \mathrm{J}'_s) \qquad \frac{1}{w_s}\right]^{b_1} \ldots \left[\frac{1}{2} R_s (\mathrm{I}_r j_s^m + q i_r \mathrm{J}_s^m) \qquad \frac{1}{w_s}\right]^{b_m}\right\}$$

$$\cos(w_r + \Sigma_1^b q w_s)$$

$$\delta_n V = \Sigma \mathrm{J}'_r \frac{1}{w_p} R_r \frac{1}{\underline{|b_1} \ldots \underline{|b_m}} \qquad (28)$$

$$\left\{\left[\frac{1}{2} R_s (\mathrm{I}_r j'_s + q i_r \mathrm{J}_s + [\mathrm{I}]\, j'_s) \frac{1}{w_s}\right]^{b_1} \ldots \left[\frac{1}{2} R_s (\mathrm{I}_r j_s^m + q i_r \mathrm{J}_s^m + [\mathrm{I}]\, j_s^m) \frac{1}{w_s}\right]^{b_m}\right\}$$

$$\sin(w_r + \Sigma_1^b q w_s),$$

with the conditions that the constants are to take in succession all values which satisfy the equations

$$\begin{aligned} w_r + q' w_{s_1} + q'' w_{s_2} + \ldots + q^b w_{s_b} &= w_p \\ b_1 + b_2 + b_3 + \ldots + b_m &= b \\ b_1 + 2b_2 + 3b_3 + \ldots + m b_m &= n - 1. \end{aligned} \qquad (29)$$

These expressions may seem to be complicated, but this is inseparable from the terms being given for the nth order of the dis-

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turbing forces. As far as the fifth power of the disturbing forces or m^{11} of the ordinary notation, they may be written down in an extremely simple form, and even to the tenth order, or m^{21}, they are neither complicated nor lengthy. The different combinations which can be taken by the suffixes $r, s_1, \ldots s_n$ will be identical with the combinations of the suffixes m in the value of the coefficient of the term with the argument w_p, in the function $\{R\}^b$ or bth power of R.

In this manner, therefore, the complete value of the perturbations of the elements of the orbit may be determined to the nth order of the disturbing forces, and this forms a complete solution of the problem of disturbed elliptic motion. The expressions which have been obtained for the perturbations admit of being transformed into still more advantageous forms. It is necessary, however, to defer these extensions and developments to a place in a second and more elaborate communication. It may be mentioned, however, that by these transformations the complete value of the perturbations of the elements may be made to assume the following form:—

$$
\begin{aligned}
v^{-1}\,\delta v &= \Big([\mathrm{P}]^p_1 R_p + \{[\mathrm{P}]^r_1\,[\mathrm{P}']^r_{s_1}\} R_r R_{s_1} + \ldots \\
&\qquad + \{[\mathrm{P}]^r_1\,([\mathrm{P}^n]^r_{s_1} + \ldots + [\mathrm{P}^n]^r_{s_n})^n\}\,(R_r R_{s_1} \ldots R_{s_n}) + \ldots\Big) \cos w_p \\
\delta V &= \Big([\mathrm{Q}]^p_1 R_p + \{[\mathrm{Q}]^p_1\,[\mathrm{Q}']^r_{s_1}\} R_r R_{s_1} + \ldots \\
&\qquad + \{[\mathrm{Q}]^r_1\,([\mathrm{Q}']^r_{s_1} + \ldots + [\mathrm{Q}^n]^r_{s_n})^n\}\,(R_r R_{s_1} \ldots R_{s_n}) + \ldots\Big) \sin w_p.
\end{aligned}
\tag{30}
$$

In this expression R_p, R_r, R_{s_1}, $\ldots R_{sn}$ are the coefficients of the arguments w_p, w_r, $w_{s_1} \ldots w_{s_n}$ in the ordinary disturbing function. The functions denoted by $[\mathrm{P}]^r_{s_m}$ and $[\mathrm{Q}]^r_{s_m}$ are known functions of the coefficients I_r, I_{s_m}, i_r, $i_{s_m} \ldots$ and the integrating factors $w_{s_m} \ldots$ They may be transformed into explicit functions of the form

$$
\phi\left\{(\mathrm{I}_r i_{s_m} + q i_r \mathrm{I}_{s_m} \qquad\qquad)\frac{1}{v}\,\frac{1}{w_{s_m}}\right\} = [\mathrm{P}]^r_{s_m}
$$

$$
\psi\left\{(\mathrm{I}_r i_{s_m} + q i_r \mathrm{I}_{s_m} + (\mathrm{I})\; i_{s_m})\,\frac{1}{v}\,\frac{1}{w_{s_m}}\right\} = [\mathrm{Q}]^r_{s_m},
$$

only under this form the terms depending on high powers of the disturbing forces are rendered more complex and lengthy.

No restriction has been placed on the number of variables represented by v and V. In the expression for the general term of R it has been assumed that they will be only six in number, or x, y, z, X, Y, Z. These are the number required by the lunar theory. The method can be applied without alteration to the planetary theory, when there will be twelve variables,

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x, x', y, y', z, z', X, X', Y, Y', Z, Z', the accented quantities referring to the disturbing planet. It can be applied, however, to any number of such variables, and absolutely without alteration.

In the preceding investigation, instead of employing the ordinary elliptic elements, use has been made of the six undefined elements whose mean values are x, y, z, X, Y, Z, and no restriction has been placed on them beyond the single condition that they are to be such as will yield a canonical system of differential equations. The perturbations of these elements have been completely determined in explicit series of functions of the form

$$R_r = (\Lambda)\, x^{\mathrm{F}_r} y^{\mathrm{G}_r} z^{\mathrm{H}}\ ,$$

$$w_s = (\lambda) + f_s X + g_s Y + h_s Z, \qquad (31)$$

$$(\mathrm{I}_r i_s + q i_r \mathrm{I}_s)\,\frac{1}{v} = (\mathrm{F}_r f_s + q\mathrm{F}_s f_r)\,\frac{1}{x} + (\mathrm{G}_r g_s + q\mathrm{G}_s g_r)\,\frac{1}{y} + (\mathrm{H}_r h_s + q\mathrm{H}_s h_r)\,\frac{1}{z}.$$

The value of the disturbing function has been already determined by me to the ninth order in terms of the usual elements of the elliptic orbit. Two courses can be adopted to render this result available for the purpose of the present method; either it can be transformed into an expression in terms of the six elements, x, y, z, X, Y, Z, or, *vice versâ*, the expression (31) can be transformed into terms of the usual elements.

Suppose

$$X + Y + Z = \mathrm{L} + \mathrm{A} + \mathrm{B} + \text{constant} = w_r,$$

where L, A, B denote the mean longitude of the Moon and the position of the perigee and ascending node of the lunar orbit. Then, as long as this condition is maintained, the arguments of the terms in the disturbing function will remain unaltered, with the exception that the coefficients f_r, g_r, h_r will be transformed into other particular values f_k, g_k, h_k.

Similarly the coefficient of the term with the argument w_r in the ordinary disturbing function may be written in the form

$$b_1 a^{\mathrm{F}_k} e^{\mathrm{G}_k} \eta^{\mathrm{H}_k} (1 + b_2 a^2 + b_3 e^2 + b_4 \eta^2 + \ldots),$$

where F_k, G_k, H_k are numerical constants, and assuming

$$x = \phi_1(a, e, \eta), \qquad y = \phi_2(a, e, \eta), \qquad z = \phi_3(a, e, \eta),$$

the preceding may be transformed into the value

$$c_1 x^{\mathrm{F}_r} y^{\mathrm{G}_r} z^{\mathrm{H}_r} (1 + c_2 x^2 + c_3 y^2 + c_4 z^2 + \ldots);$$

the only difference being that F_r, G_r, H_r will be different integers to F_k, G_k, H_k.

It is evident, therefore, that by properly defining x, y, z, either the usual disturbing function can be transformed into

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explicit functions of X, Y, Z, x, y, z, or the equations (31) transformed into explicit functions a, e, η, L, A, B, and their coefficients of F_k, G_k, H_k, f_k, g_k, h_k.

It may be remarked that the preceding method has been already employed to deduce some remarkably interesting and important theorems with regard to the terms of long period in the mean longitude. From their form it is obvious that any single term or set of terms can be determined by themselves, so that the method is peculiarly well adapted for determining the terms of long period which converge slowly and require the fourth and fifth powers of the disturbing forces to be taken into consideration. In a subsequent communication I hope to be able to deduce by its aid some important theorems with regard to the secular inequalities in the motion of the planets depending on the third and fifth powers of the disturbing forces.

London,
December 9, 1878.

Note on the Presence of Particles of Meteoric Dust in the Atmosphere.

By A. C. Ranyard, Esq.

In a Paper* read before the British Association in 1852, Professor Andrews announced that he had discovered particles of native iron in the basalt of the Giant's Causeway. Having reduced portions of the rock in a porcelain mortar to a tolerably fine powder, magnetic particles were collected by passing a magnet several times through the powder. The particles adhering to the magnet were then placed under the microscope and moistened with an acid solution of sulphate of copper. On some of them copper was deposited in a manner which indicated the presence of native iron. Professor Andrews appears to have suggested that the particles of native iron may have been derived from meteors which fell when the basalt was in a plastic condition.

In 1867 Dr. T. L. Phipson published a book entitled *Meteors, Aerolites, and Falling Stars*, in which he states† that he had frequently exposed to the wind a sheet of glass covered with some transparent mucilaginous substance in order to catch the particles of dust floating in the air. He says: "I have found that when a glass covered with pure glycerine is exposed to a strong wind late in November, it receives a certain number of *black angular particles*, some three or four of which may be thus collected in the space of a couple of hours. The experiment being made far in the country, away from the 'smuts' of a town, the black particles show themselves all the same. They are, however, not soot or charcoal; they can be dissolved in strong

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* *Brit. Assoc. Reports* for 1852. Part II. pp. 34, 35. † *See* pp. 229, 230.